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How Measurement Information Depends on Time

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Abstract

This paper contains a quantitative description of the influence of the time factor upon the information derivable from measurement. The magnitude of this information is given by the amount of information contained in the measuring instrument acting on the measured object. This amount of information is determined as a function of measurement duration in classical and quantum-mechanical cases. It is shown, under certain assumptions about the measuring process, that the measuring instrument contains no information about the measured object, when the measurement duration becomes zero.

1. *Introduction*

Physical measurement as a basic way of getting knowledge of physical objects has some interesting aspects:

(*a*) The technical aspect, relating to its technical realisation.
(*b*) The theoretical aspect, treating the theoretical problems of measuring process as, for example, compatible or incompatible measurement of two observables.
(*c*) The informational aspect, studying the measurement as a process of informational gaining.

All the above aspects of measurement need to be considered in a complete description of the measuring process. Aspect (c), however, in comparison with the earlier ones, was theoretically investigated only recently. The reason for this was mainly due to the lack of necessary conceptual and mathematical formalism, by means of which it might be possible to describe this aspect in a quantitative way. Only when the mathematical apparatus of information theory had been built up in its complete form, was it possible to turn to such a description. By this time it was necessary, however, to introduce new notions into the theory of measurement and to redefine some old notions previously used for qualitative description of the information aspect of the measuring process (e.g. 'volume of knowledge', 'information quantity,' etc.). All these concepts are exactly quantitatively defined only within the framework of information theory.

The main merit of the application of the concepts of information theory here consists in the fact that this makes it possible to determine quantitatively the information produced in a measurement, using as a measure the amount of information contained in the measuring instrument about the measured object, or more precisely, about the random variables (observable) defined on physical states of the measured object. Abstraction from concrete realisations of the measuring instruments or the measured objects, respectively, when determining the informational results of measurements, is very useful, since it allows one to concentrate on certain features of the informational aspect of measurement. This abstraction, of course, may not be suitable for studying other aspects of measurement. The influence of time factor on the informational results of a measuring process will be dealt with in this paper, by using the so-called entropic model of physical measurement (Majerník, 1968), the main features of which are as follows (Majerník, 1969a).

The measuring instrument is considered to be a probability system which for a measurement can assume a state M_i (given, for instance, by a pointer position) from a set of a measuring instrument M. On the set M a random variable is defined which will be denoted by τ_m. The measured object is represented by a probability system assuming one of its physical states, on which a physical random variable (observable) τ_0 is defined. During the measurement a statistical linkage (Majerník, 1969b) between the random variables τ_m and τ_0 is established. The magnitude of statistical linkage between the random variable τ_0 and τ_m expresses the extent of their statistical dependence, and its scalar measure is called information (Majerník, 1969b). The so-called information–entropic interaction occurs between both systems, creating a suitable statistical linkage between the random variables of the measured object and the measuring instrument. The magnitude of this information–entropic interaction is determined by the amount of information contained in the random variables τ_m about the random variable τ_0. The formation of this statistical linkage represents the necessary condition for performing the measuring process. With the information–entropic interaction physical interactions may also be generally linked, and these may sometimes interfere with the physical situation of the measured object, so that the measuring results may not correspond to the actual values of measured observables. This interference may be often quantitatively described, allowing one to find a type of information–entropic interaction which, at maximum information gain, distorts the physical situation of the measured object in a minimum way.

One of the important factors affecting the magnitude of statistical linkage between the measuring instrument and the measured object is the duration length of the information–entropic interaction, considered to be equal to the measurement duration. The time factor will be manifested in a classical physical measurement in another way than at measurement of quantum-mechanical objects. While in the first case mainly the discrimination ability of measuring instruments is changed with the length of information-entropic interaction, for quantum-mechanical measured objects the spectral decomposition of the wave function in respect of the eigenvalues of the operator coordinated to the measured observable during measurement is relevant.

Now, recall some basic concepts of information theory. In information theory, the amount of information (i.e. the magnitude of statistical linkage) is defined by means of the general integral (Kolmogoroff, 1957)

$$I(\tilde{x},\tilde{y}) = \int_X \int_Y P_{xy}(dx\,dy) \log \frac{P_{xy}(dx\,dy)}{P_x(dx)\,P_y(dy)} \quad (1.1)$$

where X and Y are sets of values of the random variables $\tilde{x}$ and $\tilde{y}$ respectively, on which the elements of the probability distributions,

$$P_x(x_i) = P(x_i \in X)$$
$$P_y(y_i) = P(y_i \in Y)$$

are given, and

$$P_{xy}(z_{ij}) = P(x_i, y_i \in Z), \qquad i, j, = 1, 2, \ldots, n$$

represents the elements of the joint probability distribution on the set of all ordered couples $z_{ij} = (x_i, y_j)$, $x_i \in X$, $y_j \in Y$. When the probability distributions are absolutely continuous, they can also be expressed by means of functions of probability density $p(x)$, $q(y)$ and $p(x,y)$. In this case, the general integral (1.1) turns out to be a Riemann one, having the form

$$I(\tilde{x},\tilde{y}) = \int_Y \int_X p(x,y) \log \frac{p(x,y)}{p(x)q(y)}\, dx\, dy \quad (1.2)$$

if $\tilde{x}$ or $\tilde{y}$ assume only discrete values, $x_1, x_2, \ldots, x_n$, or $y_1, y_2, \ldots, y_n$, with the probability distributions being

$$\mathcal{P}_x = [P(x_1), P(x_2), \ldots, P(x_n)] \qquad \text{and} \qquad P_y = [P(y_1), P(y_2), \ldots, P(y_n)]$$

then from equation (1.1)

$$I(\tilde{x},\tilde{y}) = \sum_i \sum_j P(x_i, y_j) \log \frac{P(x_i, y_j)}{P(x_i).P(y_j)} \quad (1.3)$$

where $P(x_i, y_j)$ is the probability that the random variables $\tilde{x}$ and $\tilde{y}$ simultaneously assume the values x_i and y_j respectively. The statistical linkage is mathematically expressed by means of an ensemble of conditional probabilities forming the transfer matrix R, defined as (Frey, 1963)

$$R = \begin{pmatrix} r_1(1) & \cdots & r_1(n) \\ \vdots & & \\ r_n(1) & \cdots & r_n(n) \end{pmatrix}$$

where $r_j(k)$ is the conditional probability that the random variable $\tilde{y}$ assumes its kth value, if the random variable $\tilde{x}$ takes on its jth value. Using the relation between probability distributions $P(x_j,y_k)$ and the conditional probability distribution $r_j(k)$, the expression (1.3) may be rewritten in the form

$$I(\tilde{x},\tilde{y}) = \sum_i \sum_j P(x_j)\, r_j(k) \log \frac{r_j(k)}{P(y_j)}$$

Similarly, the integral (1.2) takes the form

$$I(\tilde{x},\tilde{y}) = \int_X \int_Y p(x)\, r_x(y) \log \frac{r_x(y)}{q(y)}\, dx\, dy \qquad (1.3a)$$

with

$$q(y) = \int_X p(x)\, r_x(y)\, dx$$

where $r_x(y)$ is the density function for conditional probabilities (transfer function).

By information–entropic interaction of the measurement the aim is to create an optimal statistical linkage between the measuring instrument and measured object. This intention is, however, necessarily linked with the fact that when one is creating a statistical linkage between τ_m and τ_0, the probability distribution on the set of physical states forming its definition set is generally changing. This change of probability distribution mostly causes a change of the total physical situation of the measured object. Since the aim of a measurement is to find the actual values of the physical quantities of the original measured object, this change is most undesirable. Hence, we have the following situation. If the measurement is to be possible at all, a statistical linkage between the random variables τ_m and τ_0 must be carried out. On the other hand, however, the formation of such a statistical linkage reacts upon the original physical situation of the measured object, changing its measured data. In the optimal measurement case one obtains the maximum information for the minimum change of measured object. From a set of real physical measuring procedures it is possible to sort out those which are able to form the required optimum statistical linkage between the variables τ_m and τ_0. To describe the influence of a measuring instrument on the physical situation of a measured object is, to a certain extent, possible, due to the properties of the physical carrier of the information–entropic interaction between the measuring instrument and the object being measured.

The topics dealt with in subsequent sections will include a theoretical investigation of the influence of measurement duration upon the magnitude of statistical linkage (information) between the measuring instrument and measured object, with regard to the physical nature of the physical carrier of information–entropic interaction.

2. *The Character of Information–Entropic Interaction Associated with Classical Measurements*

We shall understand as a classical measurement, a measurement taking place between a measuring instrument and measured object for physical systems large enough to be described by laws of classical physics. A classical measuring instrument generally has a pointer as the output, and this pointer, after the measuring process, can be situated in different positions, forming in this way a metric scale (Pfanzagl, 1959). From the set of pointer positions of the measuring instrument, the random variable τ_m is defined. The values of the random variable τ_m are given by the scale values of a measuring instrument coordinated to its pointer positions. The pointer of a measuring instrument also becomes an indicator of the physical state of this instrument, which it reaches because of the information–entropic interaction with the measured object. In principle, classical measuring instruments may be divided up into two groups:

(1) Physical measuring instruments which come immediately into contact with the physical measured object. For instance, thermometers, some measuring instruments of electrostatical quantities, and so on.

(2) Physical measuring instruments whose contacts with the measured objects are intermediated by a sequence of certain signals, the sources of which represent directly the measured objects or, alternatively, interact for a time with the measured objects.

I shall treat only the first group of physical measuring instruments which can be, in principle, simulated by the following model. Let the measured object assume the ith state of physical states $S_1, S_2, \ldots, S_n$ with a certain probability P_i. The values $\mu_1, \mu_2, \ldots, \mu_n$ of the measured physical quantity are coordinated to the states $S_1, S_2, \ldots, S_n$ according to the scheme:

S	S_1	S_2	$\cdots$	S_N
$\mathcal{P}$	P_1	P_2	$\cdots$	P_N
μ	μ_1	μ_2	$\cdots$	μ_N

In what follows, let us assume that the state variable μ is related to the physical set quantity q according to the following equation (as for instance, would be the relationship between temperature and amount of heat)

$$q = \mathscr{K} . \mu \tag{2.1}$$

Let the same relation as (2.1) be also valid at the measuring instrument. To put this another way, assume that the quantity q forms an completely additive set function on a σ-algebra of subsets of the set of all measured objects coming under consideration. The values of the state variable $\mu^{(p)}$

as well as of the physical set quantity q [see equation (2.1)], can also be related to each state of the measuring instrument

$$q^{(p)} = \mathcal{K}^{(p)} \mu^{(p)}$$

The physical carrier of information-entropic interaction between the measured object and the measuring instrument is the physical quantity q. Let it have the following properties:

(1) Its total magnitude is conserved, i.e. for the system comprising measuring instrument and measured object, the following relation holds

$$q = q^{(o)} + q^{(p)} = \text{const.}$$

(2) Between the measured object and the measuring instrument, when in contact, a certain amount of the quantity q can transfer. Assume that the amount Δq passing from the measured object to the measuring instrument, and vice versa, is directly proportional to the difference between the state quantities and the duration of the information-entropic interaction T:

$$\Delta q = \lambda(\mu^{(o)} - \mu^{(p)}).T, \qquad \lambda \equiv \text{const.}$$

Let the accuracy of the measuring instrument be $\Delta\mu^{(p)}$ and let the measuring instrument have at the beginning of each measurement a position corresponding to the value of the state quantity $\tau^{(p)} = 0$. Further, the measured object should be sufficiently large that its state variable is only slightly changed as q changes. Under these conditions, the time differential of the state variable $\mu^{(p)}$ is given by the equation

$$\frac{d\mu^{(p)}}{dt} = \lambda(\mu^{(o)} - \mu^{(p)})$$

whose solution has the form

$$\mu^{(p)} = \mu^{(o)}.[1 - \exp(-\lambda T)]$$

The discrimination ability $\Delta\mu^{(p)}$ of the measuring instrument is projected onto the measured object according to the relation

$$\Delta\mu^{(o)} = \mu_2^{(o)} - \mu_1^{(o)} = \frac{\Delta\mu^{(p)}}{1 - \exp(-\lambda T)} \tag{2.2}$$

so that the number of different states of the variable $\mu^{(o)}$ for the measured object, if the total number (or extent) is given by R, is given by the equation

$$n = \frac{R.[1 - \exp(-\lambda T)]}{\Delta\mu^{(p)}} \tag{2.2a}$$

The number n represents an important characteristic of classical measurement, since it enables one to determine the magnitude of the statistical linkage between the measured object and the measuring instrument.

Physical states of the measured object have generally different probabilities. Next we shall, for the sake of simplicity, assume their probability distribution to be uniform and to have probability

$$P_i = \frac{1}{n}$$

In this case, the information can be determined by means of the simple relation

$$I(\tau_m, \tau_0) = \log n$$

hence,

$$I(\tau_m, \tau_0) = \log \frac{R[1 - \exp(-\lambda T)]}{\Delta\mu^{(p)}} \tag{2.3}$$

When discussing this relation, two circumstances must be pointed out:

(i) Information $I(\tau_m, \tau_0)$ is a positive quantity, according to its definition. Therefore, the relation (2.3) has its meaning only for a larger measurement duration than T_1, as given by the equation

$$\frac{R[1 - \exp(-\lambda T_1)]}{\Delta\tau^{(p)}} = 1$$

(ii) In the asymptotic case, for $T \to \infty$, we get $I(\tau_m, \tau_0) = \log(R/\Delta\tau^{(p)})$, whereby the information $I(\tau_m, \tau_0)$ assumes its maximum value.

As a measure for change of the physical situation of the measured object one may take the mean value of the quantity Δq passing from the measured object to the measuring instrument during a measurement, hence

$$\Delta q = \mathscr{H}^{(p)} \frac{n}{2} \Delta\tau^{(p)}$$

Considering (2.2a), one has

$$\Delta q = \frac{\mathscr{H}^{(p)} R.[1 - \exp(-\lambda T)]}{2}$$

For the optimal measurement, the information $I(\tau_m, \tau_0)$ should be as large as possible, with Δq as small as possible, that is, the function

$$f_1 = \frac{I(\tau_m, \tau_0)}{\Delta q}$$

should get is maximum value. Since f_1 is given only as a function of the variable T, it is possible to find its extreme values, and so to determine the quantitative criteria for the optimal measurement.

Generally, for classical measurement it is necessary to consider the distribution of deviations from the correct values whose magnitudes are being determined by the Gaussian probability distribution. Since the discrimination ability $\Delta\mu^{(o)}$ is time-dependent, the dispersion σ of the deviations from correct values also has this property. Supposing a linear relation between $\Delta\mu^{(o)}$ and σ, one obtains

$$\sigma(T) = k.\Delta\mu^{(o)}(T) \tag{2.4}$$

According to (2.2), the transfer function between the measured object and the measuring instrument takes the form

$$r_x(y) = \frac{1}{\sqrt{(2\pi)}\,\sigma}\,.\exp\left\{-\frac{(y-x)^2}{2\sigma^2}\right\}$$
$$= \frac{1-\exp(-\lambda T)}{\sqrt{2(\pi)}k\sigma_0}\exp\left\{-\frac{(y-x)^2[1-\exp(-\lambda T)]^2}{2\sigma_0^2}\right\}$$

where y or x is the value of the random variable τ_m or τ_0 respectively, and

$$\sigma_0 = \lim_{T\to\infty} k\Delta\mu^{(o)}$$

If the function of probability density of the random variable τ_0 is denoted by the symbol $p(x)$, then the amount of information $I(\tau_m, \tau_0)$, as a function of the duration of the measurement T, is given, according to equation (1.3a), by the expression:

$$I(\tau_m;\tau_0,T) = \int_X\int_Y p(x)\frac{[1-\exp(-\lambda T)]}{\sqrt{(2\pi)}\,\sigma_0}\exp\left\{-\frac{(y-x)^2[1-\exp(-\lambda T)]^2}{2\sigma_0^2}\right\}$$
$$.\log\frac{\dfrac{[1-\exp(-\lambda T)]}{\sqrt{(2\pi)}\,\sigma_0}\exp\left\{-\dfrac{(y-x)^2[1-\exp(-\lambda T)]^2}{2\sigma_0^2}\right\}}{q(y)}\,dx\,dy$$

with

$$q(y) = \int_X p(x)\frac{[1-\exp(-\lambda T)]}{\sqrt{(2\pi)}\,\sigma_0}\exp\left\{-\frac{(y-x)^2[1-\exp(-\lambda T)]^2}{2\sigma_0^2}\right\}dx$$

The formula (2.5) expresses the required dependence of the magnitude of statistical linkage, as well as the density function $q(y)$ of the random variable τ_m on the measurement duration at the classical measurement. There are two special cases of equation (2.5) of importance. Firstly, the case when the measurement duration is equal to zero ($T = 0$). Then

$$I(\tau_m;\tau_0,T) = 0$$

Then the random variable τ_m is not statistically linked to the measured variable τ_0, hence, the measuring instrument does not contain any information about the measured object. Under these conditions, the measuring instrument cannot perform its function. When the magnitude of statistical linkage between the random variables τ_m and τ_0 is different from zero, the inequality $T > 0$ must hold. Secondly, consider the case when $T = \infty$. Then the expression (2.5) adopts the form

$$I(\tau_m;\tau_0,T) = \int_X \int_Y p(x) \frac{1}{\sqrt{(2\pi)}\,\sigma_0} \exp\left\{-\frac{(y-x)^2}{2\sigma_0^2}\right\}$$

$$\times \log \frac{\dfrac{1}{\sqrt{(2\pi)}\,\sigma_0} \exp\left\{-\dfrac{(y-x)^2}{2\sigma_0^2}\right\}}{q(y)}\, dx\, dy$$

with

$$q(y) = \int_X p(x) \frac{1}{\sqrt{(2\pi)}\,\sigma_0} . \exp\left\{-\frac{(y-x)^2}{2\sigma_0^2}\right\} dx$$

The magnitude of statistical linkage $I(\tau_m;\tau_0)$ in this case depends mainly upon the value of the dispersion σ_0. If $\sigma_0 \neq 0$, then $q(y) \neq p(x)$, that is the probability distributions of the quantities τ_m and τ_0 are different. If $\sigma_0 = 0$, then

$$r_x(y) = \delta(y - x)$$

and

$$q(x) = p(x)$$

Then, and only then, do the values of the random measured variable τ_0 unambiguously correlate with the values of the random variable given from positions of the measuring instrument τ_m. Under these conditions, an ideal classical measurement may occur. If $0 < \sigma$ and $0 < T < \infty$, then the information lies in the interval

$$0 < I(\tau_m;\tau_0,T) < \infty$$

The determination of the information $I(\tau_m;\tau_0,T)$ is a necessary condition for informational aspect of the measurement to be described. Equation (2.5) enables one to determine information parameter for each specific case of the *a priori* probability distribution of the physical states of the measured object, given by means of the density function $p(x)$.

The coefficient of the informational effectiveness of measurement with respect to its duration may be defined in a similar way to (2.4),

$$f_2 = \frac{I(\tau_m;\tau_0,T)}{T} \tag{2.6}$$

This gets its maximum value for a finite value of T. At this value of T, the optimal measurement with regard to the effectiveness coefficient f_2 may be performed.

3. *Time Factor for Measurement of Quantum-mechanical Systems*

Now we shall deal with the problem regarding the size of the statistical linkage $I(\tau_m, \tau_0)$ between the random variable τ_m defined on the position-set of a macro-instrument, and the measured observable defined on the set of quantum states of the measured quantum-mechanical system (how much information is contained in the variable τ_m about the observable τ_0), if this instrument is at time T in information–entropic interaction with a quantum-mechanical measured object. The magnitude of this statistical linkage represents a function of the elements of the transfer matrix. For the measuring process, these elements are generally dependent on time, and thus on the information $I(\tau_0, \tau_m)$, too. By means of quantum-mechanical laws the time-dependence of elements of the transfer matrix $R(T)$ may be found, and so also the time dependence of information $I(\tau_m, \tau_0, T)$ may be searched for. For $T \to \infty$ we get a stationary measurement case. When the duration of measurement is so small that it has a marked influence on the magnitude of information $I(\tau_m, \tau_0)$ we shall speak about the non-stationary measurement case.

We shall now turn to the problem of creation of statistical linkage between the measuring instrument and the quantum-mechanical measured object. Let the measuring instrument be linked with the measured object at time t taken from the interval $\langle -T/2, T/2 \rangle$

$$t \in \langle -T/2, T/2 \rangle$$

The carrier of information during this time interval is the wave function of the quantum-mechanical object, which will be denoted by $\Psi(t, x)$. Let the measurement be performed by means of a selective measuring instrument able to distinguish physical states of the measured micro-object, and so to determine its distribution function (in the statistical interpretation of quantum-mechanics), or to determine the probability distribution on these states (in the probability interpretation of quantum-mechanics). To obtain the T-dependent expression for the amount of information, we must determine the function of probability density given on the set S of quantum states of the measured object, as well as the transfer function between the observable τ_0 and the random variable τ_m determined using positions of a selective measuring instrument. Let the space-point where the measuring instrument will be situated have its coordinate x'. Then the wave function $\Psi(x', t)$ in time $t \in \langle -T/2, T/2 \rangle$ is the only one source of information on the micro-object. During the measuring process a coordination of the actual value q of measured observable to the value being shown on the instrument scale is taking place. We shall denote the value shown on the scale corresponding to the actual value q of the measured observable τ_0 by the symbol q', and the probability distribution on the set of positions of the measuring instrument, given in the time $t \in \langle -T/2, T/2 \rangle$, by $p'(q', x')$, where the coordinate x' is taken as a parameter. The probability distribution of the random variable τ_m is generally other than that of the measured observable τ_0 given on the set of quantum states of the measured object and determined by means of the known relation

$$p(q, x') = |a(q, x')|^2$$

with

$$a(q, x') = \int_{-\infty}^{+\infty} \Psi(x', t) . \phi(q, t, x') \, dt$$

where $\phi(q,t,x)$ is the eigenfunction (corresponding to eigenvalue q) of the operator O_1 belonging to the measured observable τ_0.

To find the transfer function between the measuring instrument and measured object, we take into account the fact that the measuring instrument interacts with the measured object only in the interval $t \in \langle -T/2, T/2 \rangle$. Hence, the wave function from the side of the measuring instrument $\phi(x,t)$ may be expressed in the form

$$\phi(x,t) = 0 \qquad \text{for } t \in \langle -\infty, -T/2 \rangle$$
$$\text{and } t \in \langle +\infty, +T/2 \rangle$$
$$\phi(x,t) = \Psi(x,t) \qquad \text{for } t \in \langle -T/2, +T/2 \rangle$$

One obtains the transfer function between the elements of a set of quantum states and a set of positions of the measuring instrument by the relation

$$r_q(q';x',T) = |b(q,q',x',T)|^2$$

where

$$b(q,q',x',T) = \int_{-T/2}^{+T/2} \phi(x',q,t).\phi(x',q',t)\,dt \qquad (3.1)$$

since the expression $|b(q,q',x',T)|^2$ gives the density function of the conditional probability that the measuring instrument assumes the pointer position belonging to the quantum state of the measured object with the quantum number q', if the measured object is in a pure quantum state, characterised by the quantum number q and by the wave eigenfunction $\phi(x',q,t)$.

The function of probability density on the pointer positions of a measuring instrument $p(x',q',T)$ is determined by the equation:

$$p'(x',q',T) = |c(x',q',T)|^2$$

where

$$c(q',x',T) = \int a(q,x').b(q,q',x',T)\,dq$$

By means of the functions $b(q',q,x',T)$, $a(q,x')$ and $c(q',x',T)$ it is possible to determine the magnitude of the statistical linkage between the random variable τ_m and the measured observable τ_0. According to equation (1.3a), we get

$$I(\tau_m;\tau_0,x',T) = \iint |a(q,x')|^2.|b(q,q',x',T)|^2.$$
$$.\log\frac{|b(q,q',x',T)|^2}{|c(q',x',T)|^2}\,dq\,dq' \qquad (3.2)$$

Equation (3.2) is the required formula for the determination of the magnitude of the statistical linkage between the random variable τ_m and a general measured observable, if the wave function of the measured micro-object is given.

4. *An Example*

To discuss Equation (3.2), take for τ_0 a specific physical observable. Look for the functions $a(q,x')$, $b(q,q',x',T)$ and $c(q,x',T)$ in the case where the observable τ_0 represents the energy of a free particle which is described by the wave function $\Psi(x,t)$. In this case, equation (3.1) takes the form

$$b(E',E,T,x') = \int_{-T/2}^{+T/2} \phi(x',t,E).\phi(x',E',t)\,dt \tag{4.1}$$

where $\phi(x',t,E)$ and $\phi(x',t,E')$ are the eigenfunctions of a free particle, and E is energy of the measured quantum-mechanical object. Taking into consideration the fact that here

$$\phi(x',t,E) = f(x',E).\exp\left\{-\frac{iEt}{\hbar}\right\}$$

where $f(x',E)$ is the eigenfunction of the Hamiltonian of a free particle at the energy E, one obtains

$$b(E,E',T,x') = \int_{-T/2}^{T/2} f(x',E).f(x',E').\exp\left\{\frac{i}{\hbar}(E-E')\right\}dt = f(x',E).$$

$$.f(x',E').\frac{2\hbar}{i(E-E')}\sin\left\{\frac{(E-E')T}{2\hbar}\right\}$$

Hence, the density function of conditional probability is determined in the following way:

$$r_E(E',T,x') = \frac{4f(x',E).f(x',E')\hbar^2}{(E-E')^2}.\sin^2\left\{\frac{(E-E')T}{2\hbar}\right\} \tag{4.2}$$

Substituting equation (4.2) into equation (3.2), one obtains

$$I(E';E,T,x') = \iint |a(E,x')|^2.\frac{4f(x',E).f(x',E)\hbar^2}{(E-E')^2}\sin^2\left\{\frac{(E-E')T}{2\hbar}\right\}.$$

$$.\log\left[\frac{\dfrac{4f(x',E).f(x',E')^2\hbar^2}{(E-E')^2}\sin^2\left\{\dfrac{(E-E')T}{2\hbar}\right\}}{|c(E',T)|^2}\right]dE\,dE', \tag{4.3}$$

where

$$a(E,x') = \int_{-\infty}^{\infty} \Psi(x',t).\phi(E,t,x')\,dt$$

and

$$c(E',T,x') = \int a(E,x').\frac{2f(x',E).f(x',E')\hbar}{i(E-E')}\sin\left\{\frac{(E-E')T}{2\hbar}\right\}dE$$

Similarly to the classical case the expression (4·3) will be discussed for two limiting cases:

(i) when the duration of measurement is zero, i.e. $T = 0$;
(ii) when the measurement lasts infinitely long, i.e. $T = \infty$.

In the first case, for the magnitude of statistical linkage (information), one has $I(\tau_m; E, 0) = 0$. The values of random variables τ_m and τ_0 are therefore statistically independent, so that the pointer position of the measuring instrument does not indicate anything about the value of the measured observable.

In the second case, when $T \to \infty$, one obtains

$$\lim_{T\to\infty} \phi(E', T) = \delta(E - E')$$

so that

$$a(E) = c(E)$$

and

$$I(E'; E, T) = -\int c(E) \log c(E)\, dE + \int c(E)\, \delta(E - E').\log \delta(E - E')\, dE' \tag{4.4}$$

The second term in equation (4.4) diverges. When $c(E) \neq \delta(E)$, then

$$\lim_{T\to\infty} I(E'; E, T) = \infty$$

Under these conditions, the random variable τ_m contains maximum information about the energy of the free particle and the statistical linkage between the measuring instrument and measured object takes its maximum value.

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